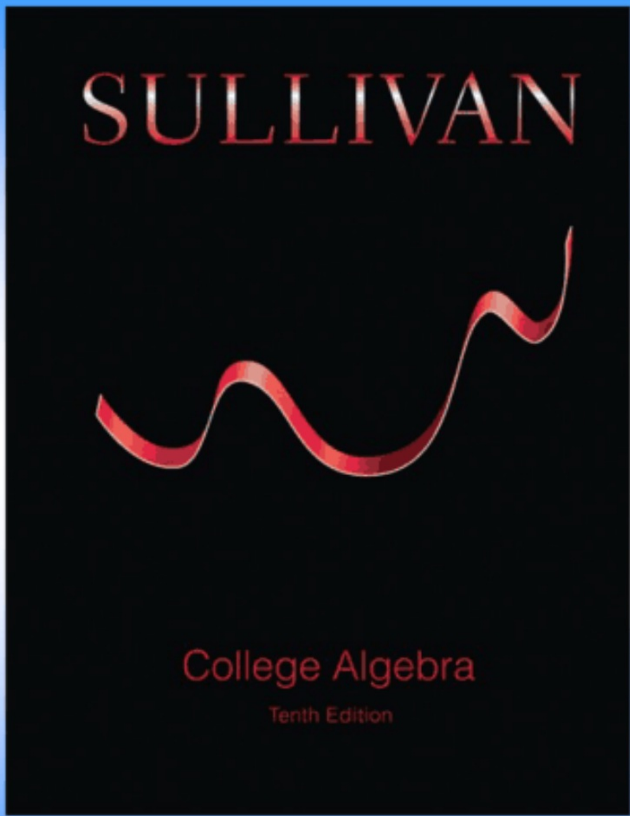
Chapter 3

Section 3

Properties of Functions



Definition

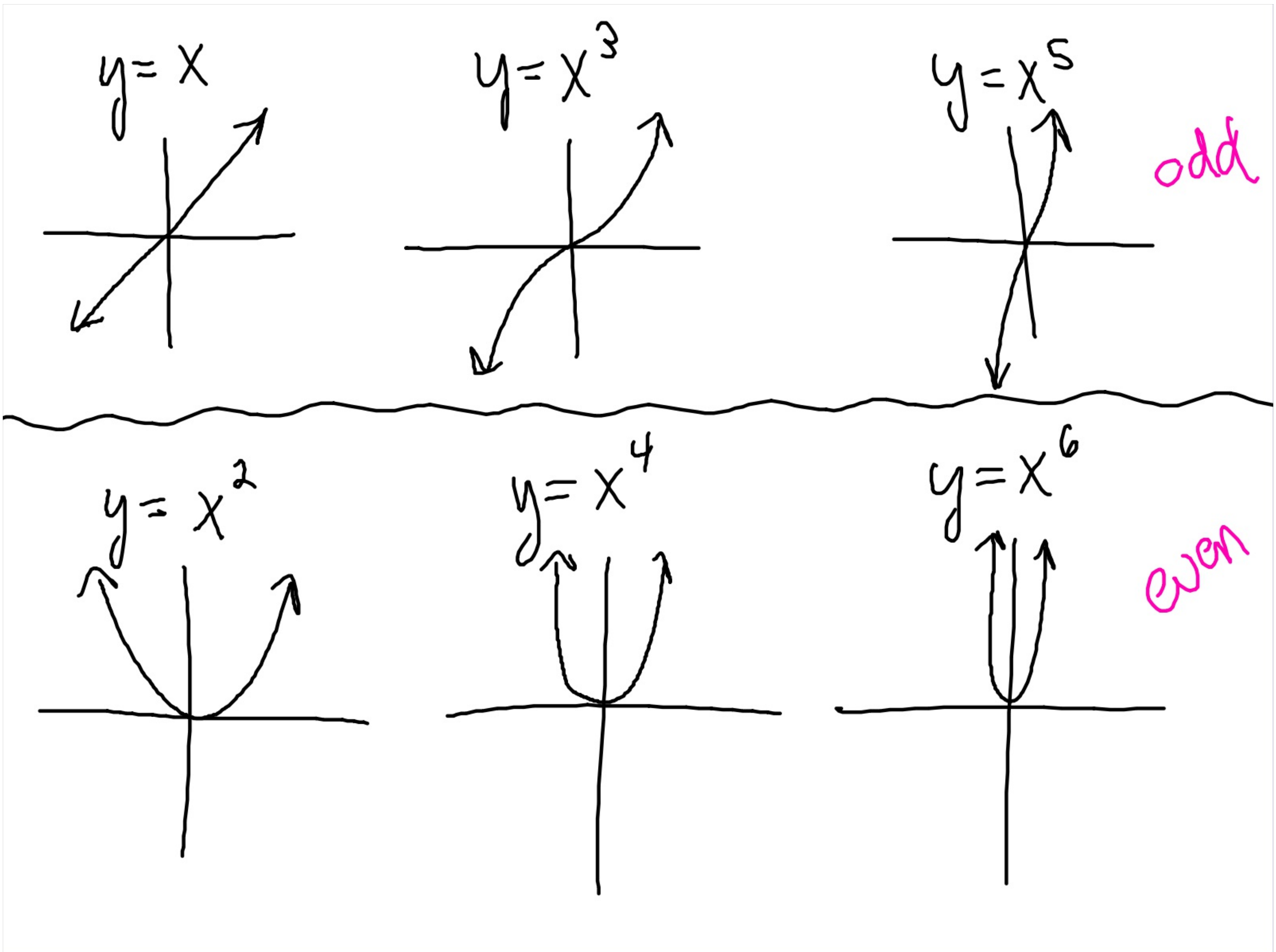
A function f is **even** if, for every number x in its domain, the number $-x$ is also in the domain and

$$\boxed{f(-x) = f(x)}$$

A function f is **odd** if, for every number x in its domain, the number $-x$ is also in the domain and

$$\boxed{f(-x) = -f(x)}$$

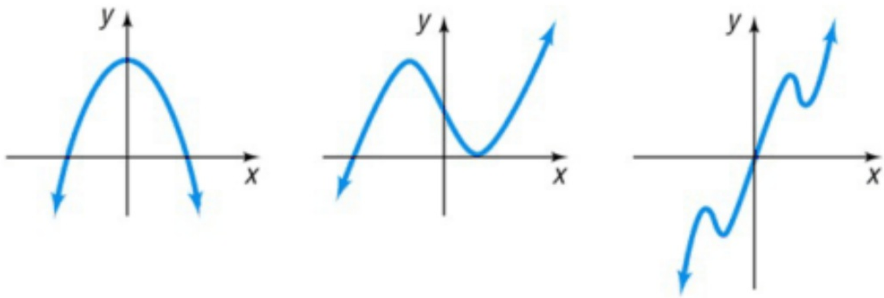
Function	Symmetric with Respect to
Even	y-axis
Odd	origin


$y = x$
$y = x^3$
$y = x^5$
odd
$y = x^2$
$y = x^4$
$y = x^6$
even

Example

Determining Even and Odd Functions from the Graph

Determine whether each graph given in Figure 17 is the graph of an even function, an odd function, or a function that is neither even nor odd.



(a) (b) (c)

Figure 17

even neither odd

Example

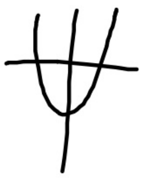
even: $(x, y) \to (-x, y)$
odd: $(x, y) \to (-x, -y)$

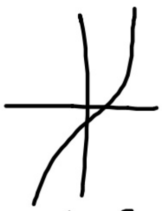
Identifying Even and Odd Functions Algebraically

Determine whether each of the following functions is even, odd, or neither. Then determine whether the graph is symmetric with respect to the *y*-axis, with respect to the origin, or neither.

(a) $f(x) = x^2 - 5$ (b) $g(x) = x^3 - 1$
(c) $h(x) = 5x^3 - x$ (d) $F(x) = |x|$

(d)  even

(a) 
even

(b) 
neither

(c) even?
$y = 5(-x)^3 - (-x)$
$y = -5x^3 + x$ ✗

odd?
$-y = 5(-x)^3 - (-x)$
$-y = -5x^3 + x$
$y = 5x^3 - x$ ✓

odd

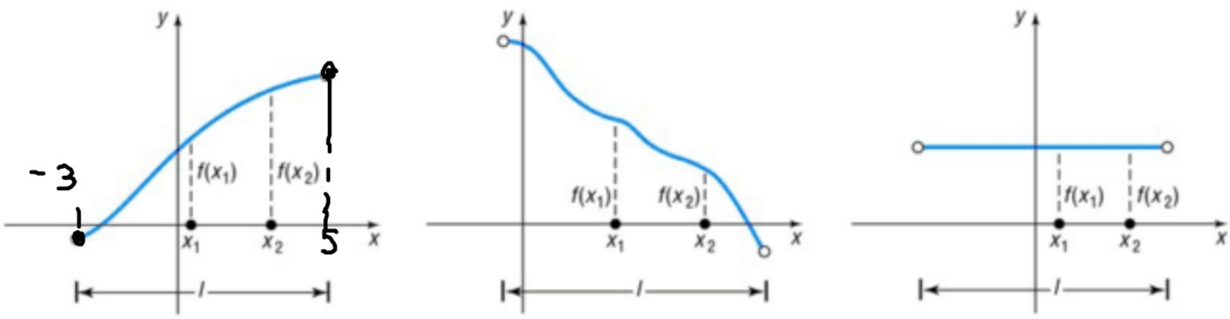
(x_1, x_2)

DEFINITIONS A function f is **increasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

A function f is **decreasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

A function f is **constant** on an open interval I if, for all choices of x in I, the values $f(x)$ are equal.

Figure: Illustration of the Definitions



(a) For $x_1 < x_2$ in I, $f(x_1) < f(x_2)$; f is increasing on I.

(b) For $x_1 < x_2$ in I, $f(x_1) > f(x_2)$; f is decreasing on I.

(c) For all x in I, the values of f are equal; f is constant on I.

increasing on $(-3,5)$

Example

Determining Where a Function Is Increasing, Decreasing, or Constant from Its Graph

Determine the values of *x* for which the function in Figure 18 is increasing. Where is it decreasing? Where is it constant?

Increasing on $(-4, 0)$

Decreasing on $(-6, -4)$ and $(3, 6)$

Constant on $(0, 3)$

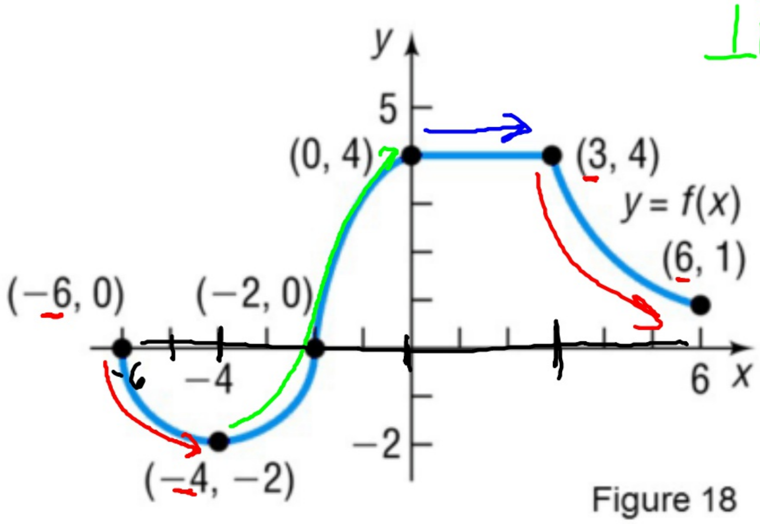
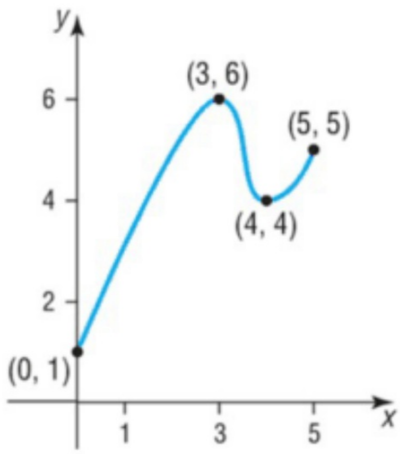


Figure 18

Your Turn

Given the graph below, determine the intervals when the function is increasing, decreasing, or constant. Also determine the domain and range.



Increasing: (0,3) and (4,5)

Decreasing: (3,4)

Constant: none

Domain: [0,5]

Range: [1,6]

Local Extrema

Let f be a function defined on some interval I.

A function f has a **local maximum** at c if there is an open interval in I containing c so that, for all x in this open interval, we have $f(x) \le f(c)$. We call $f(c)$ a **local maximum value of** f.

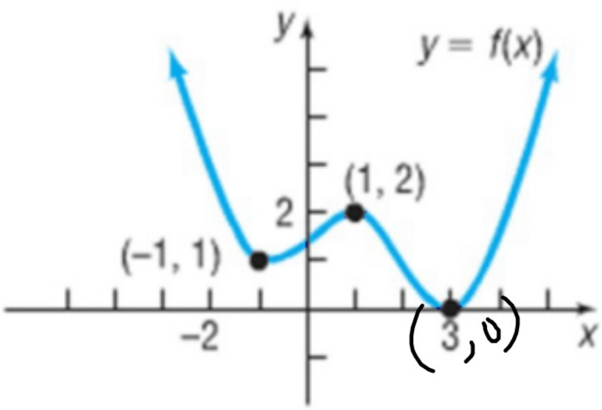
A function f has a **local minimum** at c if there is an open interval in I containing c so that, for all x in this open interval, we have $f(x) \ge f(c)$. We call $f(c)$ a **local minimum value of** f.

max
incr decr

decr incr
min

Example : Use a Graph to Locate Maxima and Minima

Using the graph below, determine the local maximum and the local minima. Also determine the domain and range.



The graph has local minima of 1,0 at $x =$ -1,3.

The graph has a local maxima of 2 at $x =$ 1.

The domain is $(-\infty,\infty)$

The range is $[0,\infty)$.

Absolute Extrema

DEFINITION Let f be a function defined on some interval I. If there is a number u in I for which $f(x) \le f(u)$ for all x in I, then f has an **absolute maximum at *u***, and the number $f(u)$ is the **absolute maximum of *f* on *I***.

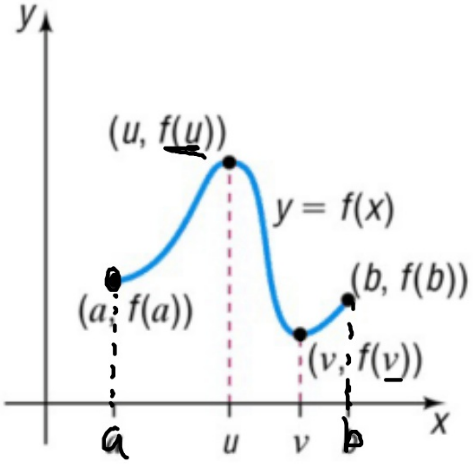
If there is a number v in I for which $f(x) \ge f(v)$ for all x in I, then f has an **absolute minimum at *v***, and the number $f(v)$ is the **absolute minimum of *f* on *I***.

Theorem

Extreme Value Theorem (EVT)

If f is a continuous function* whose domain is a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

Figure

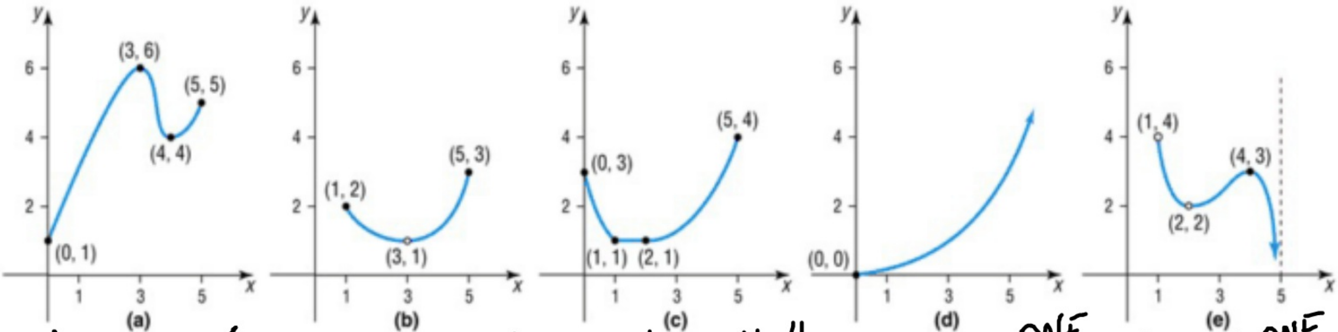


domain: $[a, b]$
for all x in $[a, b]$, $f(x) \leq f(u)$
for all x in $[a, b]$, $f(x) \geq f(v)$
absolute maximum: $f(u)$
absolute minimum: $f(v)$

Example

Finding the Absolute Maximum and the Absolute Minimum from the Graph of a Function

For each graph of a function $y = f(x)$ in Figure 23, find the absolute maximum and the absolute minimum, if they exist. Also, find any local maxima or local minima.



abs. max: 6
abs. min: 1

abs. max: 3
abs. min: DNE

abs. max: 4
abs. min: 1

abs. max: DNE
abs min: 0

abs max DNE
abs min DNE

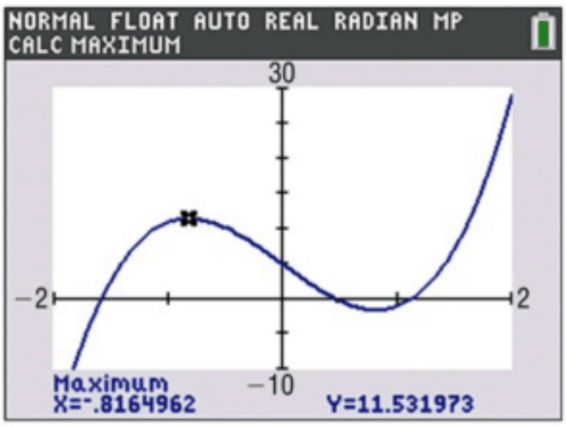
Example

Using a Graphing Utility to Approximate Local Maxima and Minima and to Determine Where a Function Is Increasing or Decreasing

(a) Use a graphing utility to graph $f(x) = 6x^3 - 12x + 5$ for $-2 < x < 2$. Approximate where f has a local maximum and where f has a local minimum.

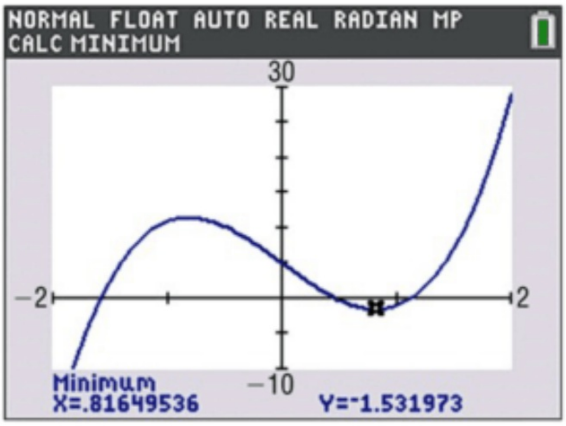
(b) Determine where f is increasing and where it is decreasing.

Solution continued



(a) Local maximum

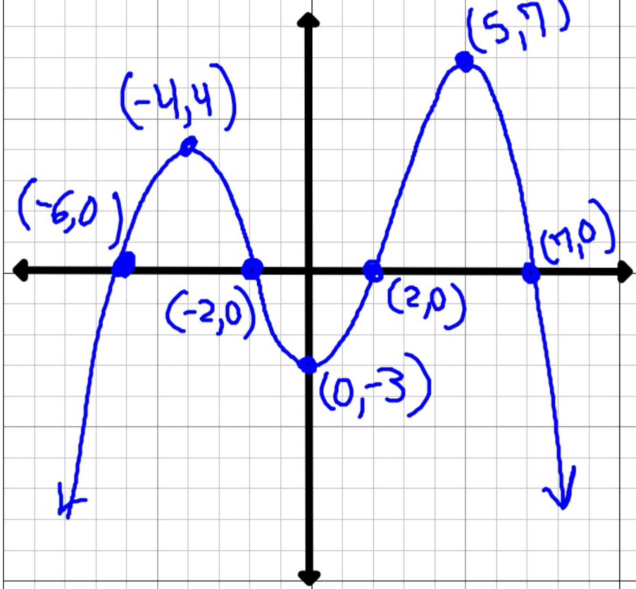
local max ≈ 11.53



(b) Local minimum ≈ −1.53

Figure 24

Increasing on $(-\infty, -0.816)$ and $(0.816, \infty)$ Decr. on $(-.8, .8)$



Local max: (5,7), (-4,4)
Local min: (0,-3)
Abs. max: 7
Abs. min: DNE

It is a function.

D: all Reals $(-\infty,\infty)$

R: $y \leq 7$ $(-\infty,7]$

Increasing on $(-\infty,-4)$ and $(0,5)$

Decreasing on $(-4,0)$ and $(5,\infty)$

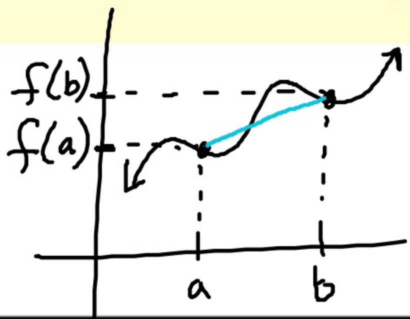
Neither even nor odd (no symmetry)

Definition

Greek "delta" Δ
means "change"

If a and b, $a \neq b$, are in the domain of a function $y = f(x)$, the **average rate of change of** f from a to b is defined as

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \qquad a \neq b \qquad \textbf{(1)}$$



Example

Finding the Average Rate of Change

Find the average rate of change of $\boxed{f(x) = 3x^2}$:

(a) From $\overset{a}{1}$ to $\overset{b}{3}$ (b) From 1 to 5 (c) From 1 to 7

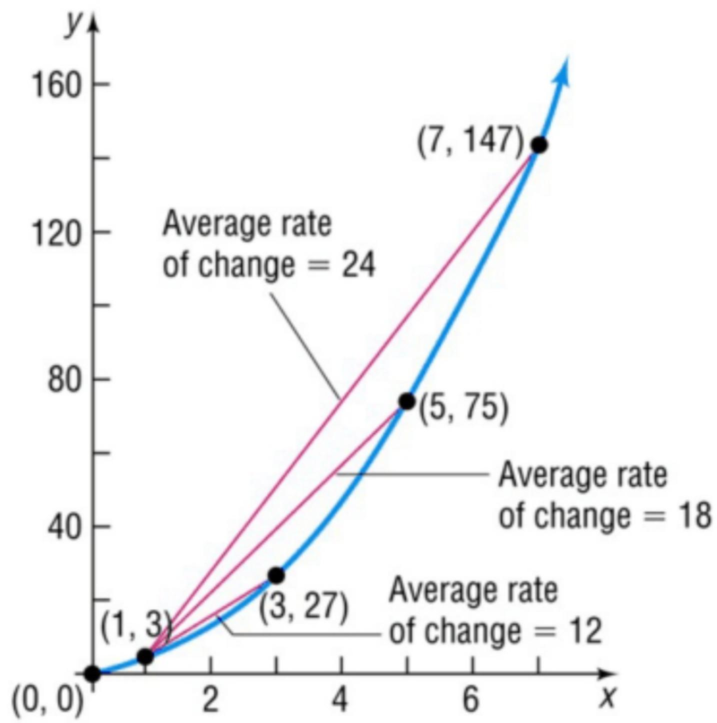
$$\frac{f(3) - f(1)}{3 - 1} = \frac{27 - 3}{2} = \frac{24}{2} = 12$$

$$\frac{f(5) - f(1)}{5 - 1} = \frac{72}{4} = 18$$

$$\frac{f(7) - f(1)}{7 - 1}$$

$$\frac{144}{6} = 24$$

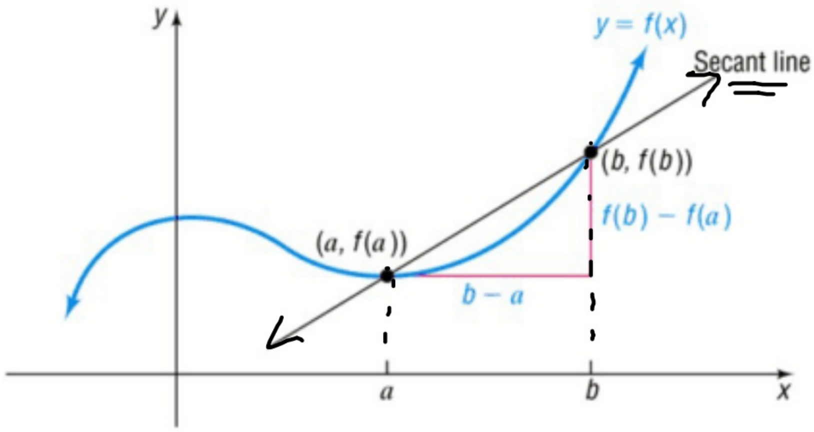
Figure: $f(x) = 3x^2$



Theorem

Slope of the Secant Line (avg. rate of change is the slope)

The average rate of change of a function from a to b equals the slope of the secant line containing the two points $(a, f(a))$ and $(b, f(b))$ on its graph.



$y = mx + b$

Example

$f(1)=1 \ldots (1,1)$
$f(-1)=7 \ldots (-1,7)$

For the function $f(x) = 4x^2 - 3x$, find the equation of the secant line from -1 to 1.

a b

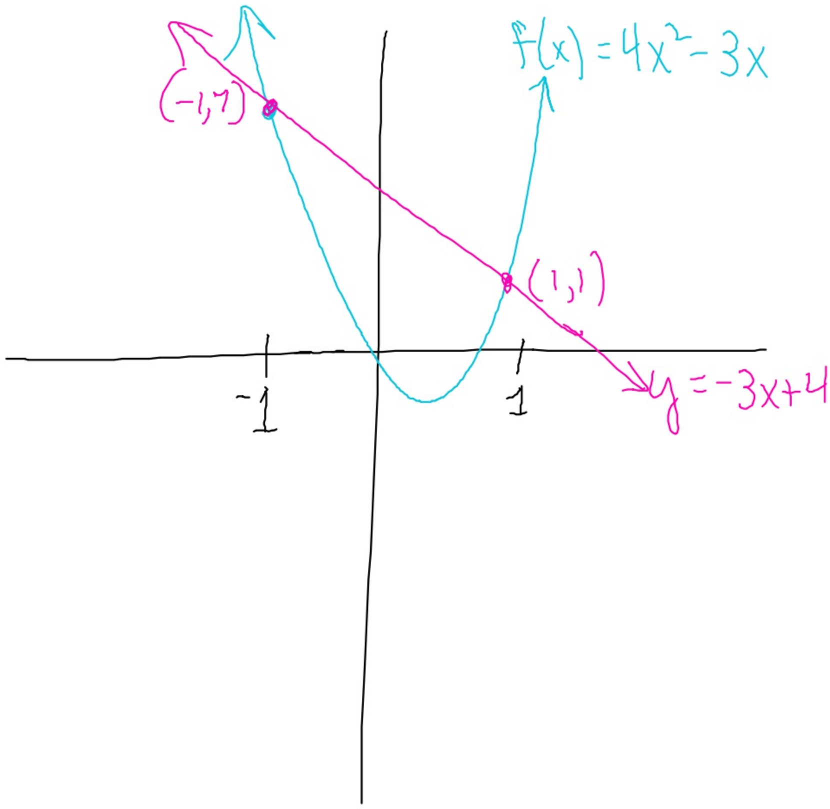
Avg. rate of change: $\dfrac{f(1)-f(-1)}{1-(-1)} = \dfrac{1-7}{2} = -3$

Equation of the secant line: $y - y_1 = m(x - x_1)$

$y - 1 = -3(x-1)$

$y - 1 = -3x + 3$

$\boxed{y = -3x + 4}$



$f(x) = 4x^2 - 3x$
$(-1, 7)$
$(1, 1)$
-1
1
$y = -3x + 4$